

Recap:

Image decomposition based on Singular Value Decomposition (SVD)

Definition: (SVD) For any $g \in M^{m \times n}$, the singular value decomposition (SVD) of g is a matrix factorization: $g = U \Sigma V^T$, where $U \in M^{m \times m}$, $V \in M^{n \times n}$ are orthogonal, $\Sigma \in M^{m \times n}$ is a diagonal matrix ($\Sigma_{ij} = 0$ if $i \neq j$) with diagonal entries given by: $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$ with $r \leq \min(m, n)$.

($U^T U = U U^T = I_{m \times m}$)
($V^T V = V V^T = I_{n \times n}$)

Remark: $r = \text{rank of } g !!$

Remark:

1. Note that $g = U \underline{\underline{\Delta}}^k V^T = \sum_{i=1}^r \sigma_i u \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}^{i\text{th}} V^T = \sum_{i=1}^r \sigma_i \vec{u}_i \vec{v}_i^T$
 $\vec{u}_i \vec{v}_i^T$ is called the eigen-image of g under SVD.

2. For $N \times N$ image, the required storage is:

$$\left(\underbrace{N}_{\vec{u}_i} + \underbrace{N}_{\vec{v}_i} + \underbrace{1}_{\sigma_i} \right) \times \underbrace{r}_{r \text{ terms}} = (2N+1)r$$

Definition: For any k ($0 \leq k \leq r$), we define

$$g_k = \sum_{j=1}^k \sigma_j \vec{u}_j \vec{v}_j^T \quad (\text{rank-}k \text{ approximation of } g)$$

Error of the approximation by SVD

Theorem: Let $f = \sum_{j=1}^r \sigma_j \vec{u}_j \vec{v}_j^T$ be the SVD of a $M \times N$ image f . For any $k < r$,

and $f_k = \sum_{j=1}^k \sigma_j \vec{u}_j \vec{v}_j^T$, we have: $\|f - f_k\|_F^2 = \sum_{i=k+1}^r \sigma_i^2$ (If $A = (a_{ij})_{1 \leq i, j \leq N}$
then $\|A\|_F^2 = \sum_{i=1}^N \sum_{j=1}^N a_{ij}^2$)

Proof: Let $f = \sum_{i=1}^r \sigma_i \vec{u}_i \vec{v}_i^T$.

Let $D \equiv f - f_k = \sum_{i=k+1}^r \sigma_i \vec{u}_i \vec{v}_i^T \in M_{M \times N}$.

Then, the m -th row, n -th col entry of D is given by:

$$D_{mn} = \sum_{i=k+1}^r \sigma_i u_{im} v_{in} \in \mathbb{R} \quad \text{where} \quad \vec{u}_i = \begin{pmatrix} u_{i1} \\ u_{i2} \\ \vdots \\ u_{im} \end{pmatrix}; \quad \vec{v}_i = \begin{pmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{in} \end{pmatrix}$$

$$\therefore D_{mn}^2 = \left(\sum_{i=k+1}^r \sigma_i u_{im} v_{in} \right)^2 = \sum_{i=k+1}^r \sigma_i^2 u_{im}^2 v_{in}^2 + 2 \sum_{i=k+1}^r \sum_{\substack{j=k+1 \\ j \neq i}}^r \sigma_i \sigma_j u_{im} v_{in} u_{jm} v_{jn}.$$

$$\begin{aligned}
\text{Thus, } \|D\|_F^2 &= \sum_m \sum_n D_{mn}^2 \\
&= \sum_m \sum_n \left(\sum_{i=k+1}^r \sigma_i^2 u_{im}^2 v_{in}^2 + 2 \sum_{i=k+1}^r \sum_{\substack{j=k+1 \\ j \neq i}}^r \sigma_i \sigma_j u_{im} v_{in} u_{jm} v_{jn} \right) \\
&= \sum_{i=k+1}^r \sigma_i^2 \sum_m u_{im}^2 \sum_n v_{in}^2 + 2 \sum_{i=k+1}^r \sum_{\substack{j=k+1 \\ j \neq i}}^r \sigma_i \sigma_j \sum_m u_{im} u_{jm} \sum_n v_{in} v_{jn} \\
&= \sum_{i=k+1}^r \sigma_i^2 = \lambda_i
\end{aligned}$$

- Remark:
- To approximate an image using SVD, arrange the eigenvalues λ_i in decreasing order and remove the last few terms in $\sum_{i=1}^r \sigma_i \vec{u}_i \vec{v}_i^T$
 - rank- k approximation is the optimal approximation using k -terms (in term of F-norm) (or with rank- k image)

How to compute SVD

Let $A \in M_{m \times n}$ ($m \geq n$) ($\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$)

Step 1: Find eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$

and orthonormal eigenvectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$
of $A^T A \in M_{n \times n}$ (with $\|\vec{v}_j\| = 1, j=1, \dots, n$) $\rightarrow V$

[Recall: $(A^T A) \vec{v}_j = \lambda_j \vec{v}_j$]

Step 2: Define: $\Sigma = \begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \dots & \\ & & & \sqrt{\lambda_n} \\ & & & & 0 \end{pmatrix} \in M_{m \times n}$
Add zero rows if $m > n$

Step 3: For non-zero $\sigma_1, \sigma_2, \dots, \sigma_r$,

$$\text{let } \vec{u}_1 = \frac{A \vec{v}_1}{\sigma_1}, \vec{u}_2 = \frac{A \vec{v}_2}{\sigma_2}, \dots, \vec{u}_r = \frac{A \vec{v}_r}{\sigma_r}$$

Step 4: Extend $\{\vec{u}_1, \dots, \vec{u}_r\}$ to the orthonormal basis
 $\{\vec{u}_1, \dots, \vec{u}_r, \dots, \vec{u}_m\}$ of \mathbb{R}^m . $\rightarrow U$

Step 5: Let:

$$U = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \dots & \vec{u}_m \\ | & | & & | \end{pmatrix} \in M_{m \times m}$$

$$V = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{pmatrix} \in M_{n \times n}$$

Then: $A = U \Sigma V^T$

Example 2.1: Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}.$$

We have

$$A^T A = \begin{pmatrix} 9 & 8 \\ 8 & 9 \end{pmatrix}. \quad (\text{Step 1})$$

Now, $\text{eig}(A^T A)$ are 17 and 1, and so $\sigma_1 = \sqrt{17}$, $\sigma_2 = 1$ and

$$\Sigma = \begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \end{pmatrix}.$$

Moreover,

$$\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This gives

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

Since

$$\sigma_1 \vec{u}_1 = A \vec{v}_1,$$

we have

$$\vec{u}_1 = \frac{1}{\sqrt{17}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{34}} \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}.$$

Handwritten notes: σ_1 under the first fraction, \vec{v}_1 under the second matrix, and $u_i = \frac{A \vec{v}_i}{\sigma_i}$ to the right.

$$\frac{A \vec{v}_2}{\sigma_2}$$

Similarly, we have

$$\vec{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The matrix U is, therefore, given by

$$U = \begin{pmatrix} \frac{3}{\sqrt{34}} & \frac{-1}{\sqrt{2}} \\ \frac{4}{\sqrt{34}} & 0 \\ \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} \end{pmatrix} \mathbf{u}_3$$

for some vector \mathbf{u}_3 orthonormal to both \mathbf{u}_1 and \mathbf{u}_2 . One possibility is

$$\vec{u}_3 = \frac{1}{\sqrt{17}} \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}.$$

Finally, the SVD of A is given by

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{34}} & \frac{-1}{\sqrt{2}} \\ \frac{4}{\sqrt{34}} & 0 \\ \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}.$$

Theorem: (Existence of SVD) Every $m \times n$ image has a SVD.

Proof: Consider the case when $m \leq n$.

We need the following theorem.

Theorem: Let $B \in M_{n \times n}$ be a real symmetric matrix. Then, \exists orthonormal eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ with corresponding eigenvalues such that

$$B = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \dots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} -\vec{v}_1^T- \\ -\vec{v}_2^T- \\ \vdots \\ -\vec{v}_n^T- \end{pmatrix}$$

Note that $gg^T \in M_{m \times m}$ and $g^Tg \in M_{n \times n}$ are symmetric.

$\therefore \exists$ n pairwise orthonormal eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ of g^Tg .

Suppose $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ are associated with non-zero eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_r$.

Note that $gg^T(g\vec{v}_i) = g(\lambda_i \vec{v}_i) = \lambda_i(g\vec{v}_i)$.

$\therefore g\vec{v}_i$ is an eigenvector of gg^T with eigenvalue λ_i .

Let $\sigma_i = \sqrt{\lambda_i}$. Then: $\|g\vec{v}_i\|^2 = (g\vec{v}_i)^T(g\vec{v}_i) = \vec{v}_i^T g^T g \vec{v}_i = \vec{v}_i^T(\lambda_i \vec{v}_i) = \lambda_i$.

$$\therefore \|g\vec{v}_i\| = \sigma_i$$

Define $\vec{u}_i = \frac{g\vec{v}_i}{\sigma_i}$. Then: $\|\vec{u}_i\| = 1$.

$$\text{Also, } \vec{u}_i \cdot \vec{u}_j = \frac{(g\vec{v}_i)^T(g\vec{v}_j)}{\sigma_i \sigma_j} = \frac{\vec{v}_i^T g^T g \vec{v}_j}{\sigma_i \sigma_j} = \frac{\lambda_j \vec{v}_i^T \vec{v}_j}{\sigma_i \sigma_j} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\vec{u}_i \cdot g\vec{v}_j = \sigma_j \vec{u}_i^T \vec{u}_j = \begin{cases} \sigma_i & \text{if } i=j \\ 0 & \text{if } i \neq j. \end{cases}$$

In matrix form, $\begin{pmatrix} -\vec{u}_1^T- \\ -\vec{u}_2^T- \\ \vdots \\ -\vec{u}_r^T- \end{pmatrix} \underbrace{g}_{m \times n} \underbrace{\begin{pmatrix} | & | & \dots & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_r \\ | & | & \dots & | \end{pmatrix}}_{n \times r} = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \dots & \\ & & & \sigma_r \end{pmatrix}$

Extend $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_r\}$ to an orthonormal basis $\{\vec{u}_1, \dots, \vec{u}_r, \vec{u}_{r+1}, \dots, \vec{u}_m\}$ of \mathbb{R}^m .

Extend $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ to an orthonormal basis $\{\vec{v}_1, \dots, \vec{v}_r, \vec{v}_{r+1}, \dots, \vec{v}_n\}$ of \mathbb{R}^n .

Then:

$$\begin{pmatrix} -\vec{u}_1^T \\ -\vec{u}_2^T \\ \vdots \\ -\vec{u}_r^T \\ \vdots \\ -\vec{u}_m^T \end{pmatrix} \underbrace{g}_{m \times n} \underbrace{\begin{pmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_r & \dots & \vec{v}_n \\ | & & | \end{pmatrix}}_{n \times n} = \underbrace{\begin{pmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_r & \\ & & & & 0 & \dots & 0 \end{pmatrix}}_{m \times n} = \underbrace{\Lambda}_{\neq}^{\frac{1}{2}}$$

(Here, we need to use the fact that $g\vec{v}_j = 0$ for $j > r$, since $\|g\vec{v}_j\| = \sigma_j = 0$ for $j > r$)

Note that $U^T U = U U^T = I$; $V^T V = V V^T = I \therefore g = U \Lambda^{\frac{1}{2}} V^T$, where

$$\Lambda = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_r & \\ & & & & 0 & \dots & 0 \end{pmatrix}$$

(The case for $m > n$ can be shown similarly)

Haar transformation

Definition: (Haar functions) The Haar functions are defined recursively as follows

$$H_0(t) \equiv \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

$$H_1(t) \equiv \begin{cases} 1 & \text{if } 0 \leq t < 1/2 \\ -1 & \text{if } 1/2 \leq t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

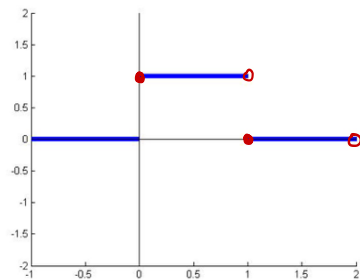
$$H_{2^p+n} \equiv \begin{cases} \sqrt{2^p} & \text{if } \frac{n}{2^p} \leq t < \frac{n+0.5}{2^p} \\ -\sqrt{2^p} & \text{if } \frac{n+0.5}{2^p} \leq t < \frac{n+1}{2^p} \\ 0 & \text{elsewhere} \end{cases}$$

where $p=1, 2, \dots$; $n=0, 1, 2, \dots, 2^p-1$

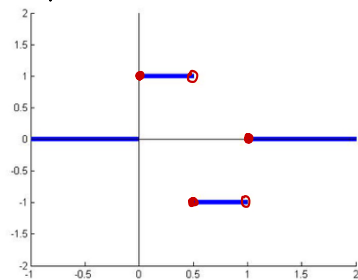
Remark: If p is larger, H_{2^p+n} is compactly supported in a smaller region.

Examples of Haar functions:

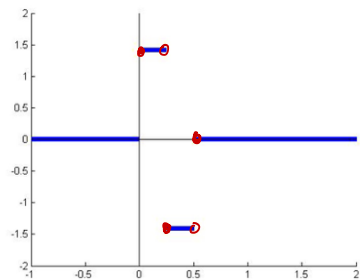
H_0



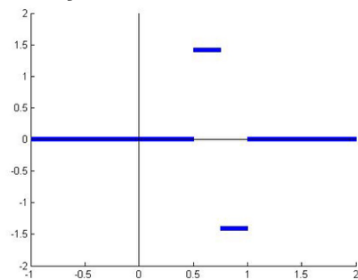
H_1



H_2

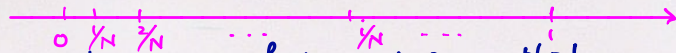


H_3



Definition (Discrete Haar Transform)

The Haar Transform of a $N \times N$ image is done by dividing $[0, 1]$ into partitions.



Let $H(k, i) \equiv H_k\left(\frac{i}{N}\right)$ where $k, i = 0, 1, 2, \dots, N-1$.

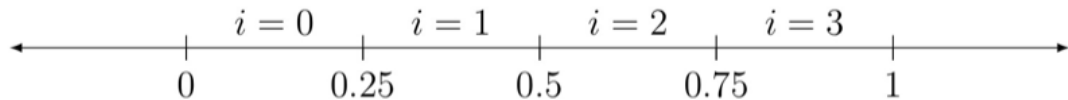
We obtain the Haar Transform matrix: $\tilde{H} \equiv \frac{1}{\sqrt{N}} H$ where $H \equiv (H(k, i))_{0 \leq k, i \leq N-1}$.

The Haar Transform of $f \in M_{N \times N}$ is defined as:

$$g = \tilde{H} f \tilde{H}^T.$$

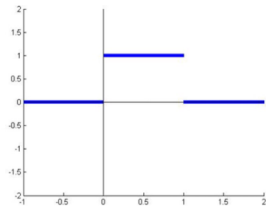
Example Compute the Haar Transform matrix for a 4×4 image.

Solution: Divide $[0, 1]$ into 4 portions:

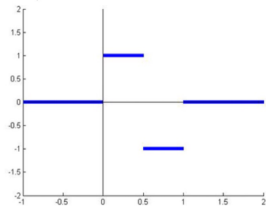


Need to check:

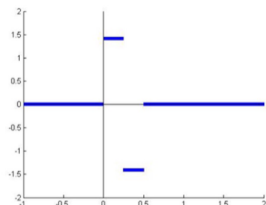
H_0



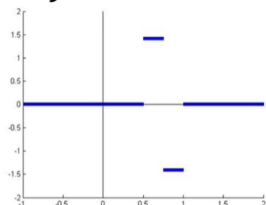
H_1



H_2



H_3



We get that:

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \quad \text{and} \quad \tilde{H} = \frac{1}{\sqrt{4}}H = \frac{1}{2}H$$

Easy to check that $\tilde{H}^T \tilde{H} = I$.